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Relating classical spinning particles to Dirac 4-spinors

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Abstract

The first problem faced is that of finding a transition from the quantum description of a fermion by means of a 4-spinor satisfying Dirac's equation to a classical or ray optics limit describing a spinning particle in an electromagnetic field. The solution is obtained by first using quaternions as a natural tool to describe the orientation of a classical particle and then transforming Dirac's 4-spinor into a quaternion in a novel way. These two quaternions are shown to be closely related and facilitate the transition between the two pictures in the ray optics approximation. The same formalism is then applied to the second problem which is comparing the exact classical and Dirac solutions to the motion of a particle in a plane electromagnetic field. Here the correspondence is effectively one of identity. This work involves using an improved classical equation of motion for a classical spinning charged particle which could be of practical value in experimental situations where the spin of a fermion is relevant but interference and other quantum effects are not. Finally there is derived the unexpected consequence that the magnitude of the intrinsic angular velocity of the classical particle is twice the mass, in the absence of an electromagnetic field, or twice the action in the more general case.

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1. Introduction

Central to this paper is the use of quaternions to form a bridge between the classical and quantum descriptions of a spinning particle. Although many authors such as Davies [1], Conway [2], Adler [3], Girard [4], Edmunds [5], Gough [6, 7], Bell *et al* [8] have used quaternions in a variety of ways in a quantum context, none has either been addressing the current problem or used the same method.

The specific initial goal addressed here is that of obtaining the classical limit of Dirac's equation in the sense that ray optics is the classical limit of Maxwell's equations. This is *not* tackled by examining the relationship between quantum spin and the spin of a classical particle. That approach goes back a long time to the work of Bargmann *et al* [9]; a recent paper giving the most recent progress and history of this approach is that of Gaioli and Alvarez [10]. Instead a totally different line is taken which is to ask what a 4-spinor can *mean* in a classical limit. This is far from obvious since Dirac produced his spinor effectively out of thin air, with no classical antecedent [5, 13]. The problem is faced in two steps.

The first step is to create a classical, relativistic description of a rotating particle. The natural description of a body rotating in three dimensions is a real quaternion. Generalizing this to relativistically moving bodies simply extends this description to quaternions with coefficients which are complex numbers: the complex quaternions. In future the word 'quaternions' will always be supposed to mean complex quaternions. With the aid of this quaternionic notation (q-notation) we can readily derive a q-equation for the motion of a rotating charged particle in an electromagnetic field. The electromagnetic field and Maxwell's equations will also be written in q-notation, as has been done elsewhere [5].

The second step involves rearranging the four complex entries of a Dirac 4-spinor in a specific way to form the four complex coefficients of a quaternion. Earlier authors, following on Conway [2], have observed that it is possible to formulate Dirac's equation in terms of quaternions rather than 4-spinors. The method used here differs from those used elsewhere because we require the resultant quaternion to transform under Lorentz transformations in the same way as the quaternion we have derived to describe the classical relativistic spinning particle. It is then possible to write Dirac's equation in terms of this new quaternion: the resultant form is different from earlier forms chosen.

The third step is then to analyse the solution of Dirac's q-equation with an eye on the high-frequency limit, in the spirit of ray optics. It is then shown that in the limit we can readily derive the classical q-equation for a relativistic spinning particle in an electromagnetic field, which achieves our first goal.

This achievement is then underlined by solving Dirac's equation exactly—not the high-frequency limit—in the special case in which a charged fermion is moving in a plane (but not necessarily periodic) electromagnetic wave. In that specific case the correspondence between the classical and quantum solutions is remarkably close, to the extent that the quantum solution is indistinguishable from a simple collection of the trajectories of classical rotating relativistic particles.

A final point of interest is the fact that the correspondence naturally throws up a measure of the intrinsic angular velocity of the rotating classical particle. This turns out to have the highly unexpected value of twice the mass—in natural units. Although no conclusions are drawn from this in this paper, it may be of value to have a purely geometric interpretation of mass when we go to the classical limit. This creates an original parallel to the idea central to general relativity which is an identity between a geometric quantity—the moment of rotation—and an energy-like quantity—the stress–energy tensor.

It is important to note that this paper only claims to deal with the interface between classical descriptions of particles and quantum or c-number field descriptions. It *excludes* therefore any discussion of quantized fields and all the complexities of particle creation and annihilation. In particular when, in later sections, solutions to the Dirac equation are cited which involve external electromagnetic fields, it is assumed in the normal way that such fields are small enough for pair creation to be insignificant, so that we can retain, in this approximation, a single particle interpretation of the solutions.

2. Notation

Since we will be working extensively with complex quaternions there is no particular reason for using Hamilton's basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, rather than the same multiplied by i, which then have the same operational properties as the familiar Pauli matrices σ^i. Other advantages of this representation include an easy transition to a world of results familiar to those in the field of quantum theory, and that some results are more easily proved by using a 2×2 matrix representation of a quaternion and so Pauli's representation will be used in this paper. On the other hand, it should be noted that Hamilton's notation can at times lead to more concise results in many contexts, cf Gough [11].

If we write a general quaternion as $Q = q^0 + q^i\sigma^i$ (with implicit summation over index i), then the following conventions will be used. If z^* denotes the complex conjugate of a complex number z, the Hermitian conjugate of Q is then $Q^\dagger = q^{0*} + q^{i*}\sigma^i$; the vector conjugate of Q is $\tilde{Q} = q^0 - q^i\sigma^i$; the scalar part of Q is $S(Q) = q^0 = \frac{1}{2}(Q + \tilde{Q})$ and the vector part is $V(Q) = q^i\sigma^i = \frac{1}{2}(Q - \tilde{Q})$. The quantity $S(P\tilde{Q}) = p^0q^0 - p^iq^i$ will be abbreviated $P \cdot Q$. NB: it is important to avoid confusing this notation with the dot or scalar product of two 3-vectors. The scalar but complex and possibly zero quantity $Q\tilde{Q}$ will be written as $\det(Q)$ rather than as $|Q|$ because this latter form suggests that it is a real quantity, which it only is for real quaternions. The notation is motivated because if we write out Q using the usual form for the Pauli matrices then the determinant is indeed $Q\tilde{Q}$. When $\det(Q)$ is non-zero then the inverse of Q is $Q^{-1} = \tilde{Q}/\det(Q)$.

At times it will be convenient for notational purposes to introduce σ^0 as the unit 2×2 matrix. Indices will not be raised or lowered on the symbols for these basis elements, but we may note the identity $\tilde{\sigma}^\mu\sigma^\nu + \tilde{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}$, the usual Lorentz metric, with $g^{00} = 1$, $g^{ii} = -1$. 4-vectors will be written as V^μ and 3-vectors will be written as $\boldsymbol{v}$. A useful identity is that

$$(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = \boldsymbol{a} \cdot \boldsymbol{b} + \mathrm{i}(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{\sigma}. \tag{1}$$

A very commonly occurring quantity is $\partial \equiv \sigma^i\partial/\partial x^i$, and it is easily shown that

$$\partial(q_0 + \boldsymbol{q} \cdot \boldsymbol{\sigma}) = \operatorname{div} \boldsymbol{q} + (\operatorname{grad} q_0 + \mathrm{i}\operatorname{curl} \boldsymbol{q}) \cdot \boldsymbol{\sigma}. \tag{2}$$

Finally, when relativistic and quantum effects are being discussed, units will be chosen so that the speed of light c and the reduced Planck's constant $\hbar$ will both be taken as unity.

3. Classical electrodynamics in q-notation

Real quaternions are to orientations and rotations what vectors are to positions and displacements. A position vector can be thought of as the displacement required to move from a reference position (the origin) to a given position. Likewise, the orientation quaternion of a body can be defined as the rotation required to turn it from some reference orientation to the given orientation. This generalizes to Lorentz space in which we will be referring to orientation in spacetime rather than in three-space.

This section is devoted to making familiar the way in which quaternions are used to describe rotating bodies (or equivalently frames of reference) in three and four dimensions. The results introduced without explanation in this section are standard and to be found in the literature [14].

The quaternion which represents a 3-rotation through an angle θ about the direction of the unit vector $\boldsymbol{e}$ is

$$R = \exp\left(-\tfrac{1}{2}\mathrm{i}\theta\boldsymbol{e} \cdot \boldsymbol{\sigma}\right) = \cos\tfrac{1}{2}\theta - \mathrm{i}\boldsymbol{e} \cdot \boldsymbol{\sigma}\sin\tfrac{1}{2}\theta. \tag{3}$$

The way in which such representations are used conveniently to describe rotations is summarized by the following equation:

$$\boldsymbol{x} \cdot \boldsymbol{\sigma} = R\boldsymbol{x}_0 \cdot \boldsymbol{\sigma}R^\dagger. \tag{4}$$

In this equation $\boldsymbol{x}_0$ is the Euclidean coordinate of any point in the rigid body (or frame of reference) in its original or reference position and $\boldsymbol{x}$ is the point to which it is moved by the rotation R, and $R^\dagger$ is the Hermitian conjugate of R.

In fact, for rotations it is the case that $R^\dagger = \tilde{R} = R^{-1}$, but the Hermitian rather than the other forms is used in the equation above for reasons to do with further extensions.

The quaternionic forms for rotations are combined by straightforward multiplication. If a rotation R_1 is followed by a rotation R_2 then the combined rotation is

$$R = R_2 R_1. \tag{5}$$

Note that both orientations and rotations are expressed in the same form, in essentially the same way as positions and displacements are written in the same vector notation. The context will indicate what is meant.

3.1. Body axes and space axes

In dealing with the kinematics of rigid bodies it is often necessary to distinguish between vectors which are defined relative to axes fixed in space and those which are relative to axes fixed in a rotating or moving body. Suppose that a body is rotating steadily about the z direction with angular velocity ω then its orientation at time t will be given by

$$R = \exp\left(-\tfrac{1}{2}\mathrm{i}\omega\sigma^3 t\right). \tag{6}$$

Now let R_0 be some constant rotation, which takes the direction of the z-axis into some other direction $\boldsymbol{e}$ (hence $R_0\sigma^3 R_0^\dagger = \boldsymbol{e}\cdot\boldsymbol{\sigma}$ by equation (4)). We then find that the orientation of the body at time t becomes

$$R(t) = R_0 \exp\left(-\tfrac{1}{2}\mathrm{i}\omega\sigma^3 t\right) = \exp\left(-\tfrac{1}{2}\,\mathrm{i}\omega\boldsymbol{e}\cdot\boldsymbol{\sigma}t\right) R_0. \tag{7}$$

This can be proved by expressing the exponentials in terms of sines and cosines, and using the fact that for rotations $R^\dagger = R^{-1}$. This equation familiarizes us with two ways of looking at the rotating body. The first form describes the orientation as a result of the steady rotation about the body axis (the z-axis) and then applies the rotation R_0 to take the axis to the observed position in space. The second form amounts to first rotating the body to the new orientation and then describes it spinning about the carried position of its spin axis. These are totally equivalent. Note then that transformations relative to body axes involve action on the right and transformations relative to space axes act on the left.

We next consider the case in which a rigid body is being rotated in a rather general way, with its orientation at any time t given by $R(t)$. Then the position of any point in the body becomes $\boldsymbol{x}(t)$ with

$$\boldsymbol{x}\cdot\boldsymbol{\sigma}(t) = R(t)\boldsymbol{x}_0\cdot\boldsymbol{\sigma}\boldsymbol{R}^\dagger(t). \tag{8}$$

Differentiating with respect to time gives

$$\dot{\boldsymbol{x}}\cdot\boldsymbol{\sigma}(t) = (\dot{R}R^{-1})\boldsymbol{x}\cdot\boldsymbol{\sigma}(t) + \boldsymbol{x}\cdot\boldsymbol{\sigma}(t)(\dot{R}R^{-1})^\dagger. \tag{9}$$

It is readily shown that for a rotation R the expression $\dot{R}R^{-1}$ has the form

$$\dot{R}R^{-1} = -\tfrac{1}{2}\,\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma} \tag{10}$$

where $\boldsymbol{\Omega}(t)$ is some real 3-vector. Using result (9) gives

$$\dot{\boldsymbol{x}}\cdot\boldsymbol{\sigma}(t) = (\boldsymbol{\Omega}(t)\wedge\boldsymbol{x}(t))\cdot\boldsymbol{\sigma} \tag{11}$$

which gives the familiar $\dot{\boldsymbol{x}}(t) = \boldsymbol{\Omega}(t)\wedge\boldsymbol{x}(t)$: the formula giving the instantaneous velocity of a point of a rigid body which is rotating with angular velocity $\boldsymbol{\Omega}(t)$. Hence, we can always use equation (10) to connect the angular velocity of a body with the first derivative of its orientation.

It will be of interest to discover the quaternionic representation of a body which is steadily precessing. This is the condition of a body which has an angular velocity composed of a component $\mathbf{\Omega}$ fixed in spatial axes and a component ω fixed in body axes.

Using the forms of equation (7) appropriate for rotations in these two ways gives straightforwardly the result that the orientation $R(t)$ is given by

$$R(t) = \exp\left(-\tfrac{1}{2}\mathrm{i}\mathbf{\Omega}\cdot\boldsymbol{\sigma}t\right) R(0) \exp\left(-\tfrac{1}{2}\mathrm{i}\omega\sigma^3 t\right) \tag{12}$$

where $R(0)$ is the orientation at $t = 0$.

As an exercise the above equation can be differentiated and equation (10) which relates the derivative of R to the angular velocity can be used to show that the total angular velocity is

$$\mathbf{\Omega}\cdot\boldsymbol{\sigma} + R(t)\omega\sigma^3 R^\dagger(t) \tag{13}$$

which, as expected, gives a component $\mathbf{\Omega}$ fixed in space axes and a component ω about the z-axis fixed in body axes.

3.2. Relativistic generalizations

If we now generalize from the rotation quaternions we have been using to more general quaternions L (for Lorentz) with complex coefficients which are no longer limited by the condition that $\tilde{L} = L^\dagger$, then we can find a generalization of 3-rotations to 4-rotations. However, L remains limited by being unimodular:

$$\tilde{L}L = 1. \tag{14}$$

The generalizations are as follows: a point $x_0 \equiv t_0 + \boldsymbol{x}_0\cdot\boldsymbol{\sigma}$ is taken by L to $x \equiv t + \boldsymbol{x}\cdot\boldsymbol{\sigma}$, where

$$x = Lx_0L^\dagger. \tag{15}$$

The form of L for a simple boost in the direction $\boldsymbol{e}$ is simply

$$B = \exp\left(\tfrac{1}{2}\chi\boldsymbol{e}\cdot\boldsymbol{\sigma}\right) \tag{16}$$

and one way of writing the most general Lorentz transform is simply

$$L = \exp\left(\tfrac{1}{2}\chi\boldsymbol{e}_2\cdot\boldsymbol{\sigma}\right)\exp\left(-\tfrac{1}{2}\mathrm{i}\omega\boldsymbol{e}_1\cdot\boldsymbol{\sigma}\right) \tag{17}$$

which is to say by first defining a 3-rotation that brings the body into a particular orientation and then applying the boost which gives it the observed speed.

A further useful result is that

$$LL^\dagger = V^0 + V^i\sigma^i \tag{18}$$

where V is the 4-velocity of the particle. If we have a point particle with charge e following trajectory $x = x(t)$ then it follows that the 4-current $J^\mu = eV^\mu\delta(x - x(\tau))$, which in q-notation is simply

$$J = eLL^\dagger\delta(x - x(\tau)). \tag{19}$$

The three-dimensional concept of angular velocity readily generalizes to four dimensions in the form of an antisymmetric tensor $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$. A body whose trajectory is subjected to an instantaneous angular velocity $\Omega^{\mu\nu}$ at each point will then have a 4-velocity satisfying the equation

$$\mathrm{d}V^\mu/\mathrm{d}\tau = -\Omega^\mu_{\ \nu}V^\nu. \tag{20}$$

Antisymmetric tensors can also be written in terms of a q-basis as

$$\Omega \equiv \tfrac{1}{2}\mathrm{i}\Omega^{\mu\nu}\sigma^\mu\tilde{\sigma}^\nu = (\Omega^{jk} + \mathrm{i}\Omega^{0i})\sigma^i \tag{21}$$

where the coefficient is chosen to reduce to the form $\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}$ for 3-rotations. Then equation (20) can be written in q-notation as

$$\dot{V} = \tfrac{1}{2}\mathrm{i}(\Omega V - V\Omega^{\dagger}) \tag{22}$$

which can be verified by expanding the right-hand side and using $\tilde{\sigma}^{\mu}\sigma^{\nu} + \tilde{\sigma}^{\nu}\sigma^{\mu} = 2g^{\mu\nu}$.

3.3. A note on transformation properties

It takes a little time when first dealing with q-notation to get out of the habit of attempting to recognize the Lorentz-transformation properties of a quantity by inspecting the indices because there are none. It is therefore worth emphasizing the following facts regarding the transformations in q-notation. A 4-orientation or 4-rotation Q transforms under a general Lorentz transformation L as follows:

$$Q \to LQ. \tag{23}$$

A 4-vector $X = X^{\mu}\sigma^{\mu}$ transforms as

$$X \to LXL^{\dagger}. \tag{24}$$

Finally a tensor $\Omega = \frac{1}{2}\mathrm{i}\Omega^{\mu\nu}\sigma^{\mu}\tilde{\sigma}^{\nu}$ can be shown to transform as

$$\Omega \to L\Omega\tilde{L} \tag{25}$$

so that if X is a vector, so is ΩX as required by (22).

These will seem a little unusual at first and arouse the objection that it is no longer possible to write down equations that are overtly covariant in the way that is possible with the more familiar index notation. While this is true, it is worth remembering that it is a general principle that different forms of notation have value for particular purposes: thus we choose rectangular Cartesian coordinates for some problems and polar coordinates for others. And it will be shown that there is a certain gain in physical insight available if we move to a quaternionic notation which balances the fact that equations do not have the familiar covariant form. Also in many cases formulae are that much simpler because of the absence of indices, as can be seen from the above formulae. It is worth adding, moreover, that with a little experience and a familiarity with the way the various types of quantities transform it is quite easy to verify Lorentz invariance in this notation.

We have earlier seen the quaternionic description of a rotating body precessing in three dimensions. This also can be generalized to a relativistically moving body, as follows. It will be defined as a body which is rotating with angular velocity ω fixed in body axes, but whose body axes are then given a rather general Lorentz transform defined relative to space axes—the generalization of the steady rotation which gave a three-dimensional rotation. For reasons to do with future applications we will consider the slightly more general case in which the magnitude of the angular velocity is a function of τ. We may also without loss of generality choose the axis of rotation to be the z-axis in body axes.

Its 4-orientation will then be given at proper time τ by a quaternion of the form

$$Q(\tau) = L(\tau)Q(0)\exp\left(-\tfrac{1}{2}\mathrm{i}\theta(\tau)\sigma^{3}\right) \qquad \text{with} \qquad L(0) = 1. \tag{26}$$

If we differentiate this with respect to τ we find that

$$\dot{Q} = \dot{L}L^{-1}Q - \tfrac{1}{2}\mathrm{i}Q\omega(\tau)\sigma^{3} \tag{27}$$

where

$$\omega(\tau) = \dot{\theta}. \tag{28}$$

But $S(\dot{L}L^{-1}) = 0$, so that $\dot{L}L^{-1} = -\frac{1}{2}\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\Omega}(\tau)$ is now some complex 3-vector. The factor $-\frac{1}{2}$ is chosen to parallel equation (10).

Hence we have the general equation

$$\dot{Q}(\tau) = -\tfrac{1}{2}\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}Q - \tfrac{1}{2}\mathrm{i}Q\omega(\tau)\sigma^3. \tag{29}$$

From this we can find the equation satisfied by the 4-velocity $V \equiv QQ^\dagger \equiv LL^\dagger$, which proves to be

$$\dot{V} = \tfrac{1}{2}\mathrm{i}(V\Omega^\dagger - \Omega V) \tag{30}$$

which is identical to equation (22) showing that $\Omega \equiv \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$ is the generalized angular velocity defined above.

3.4. Maxwell's equations

The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are combined to give a single quaternion

$$q = (-\boldsymbol{E} + \mathrm{i}\boldsymbol{H})\cdot\boldsymbol{\sigma} = \tfrac{1}{2}F^{\mu\nu}\tilde{\sigma}^\mu\sigma^\nu. \tag{31}$$

The current vector and vector potential are written as $J = J^\mu\sigma^\mu$ and $A = A^\mu\sigma^\mu$, respectively. It is then a matter of simple algebra ([6] and references cited therein) to establish that Maxwell's equations become

$$(\partial_0 - \partial)q = J. \tag{32}$$

A and q are related as

$$q = (\partial_0 + \partial)A - \partial_\mu A^\mu \tag{33}$$

so that with the Lorentz choice of gauge

$$q = (\partial_0 + \partial)A. \tag{34}$$

From equations (31) and (34), we readily obtain the familiar

$$\Box A = J. \tag{35}$$

whose appearance is essentially unchanged in q-notation.

3.5. Motion of a classical charged particle

The equations which describe the motion of particle with mass m and charge e, which may be positive or negative, in an electromagnetic field $A^\mu(x)$ can be derived from a variational principle. It is a familiar fact that if we consider any curve in spacetime given parametrically by $x^\mu = x^\mu(\alpha)$, and define the following integral of the action Γ along that curve:

$$S = \int \Gamma\,\mathrm{d}\alpha = \int \left[m\left(\frac{\mathrm{d}x_\mu}{\mathrm{d}\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\right)^{\frac{1}{2}} + eA_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\alpha}\right]\mathrm{d}\alpha \tag{36}$$

then the assumption that the classical trajectories are those for which the above integral is stationary with respect to small variations in the trajectory give rise to the familiar equation:

$$m\frac{\mathrm{d}V^\mu}{\mathrm{d}\tau} = eF^\mu_\nu V^\nu \tag{37}$$

where τ is the proper time measured along the trajectory. This can be translated into q-notation, with $V \equiv V^\mu\sigma^\mu$, as

$$m\dot{V} = -\tfrac{1}{2}e(Vq + q^\dagger V). \tag{38}$$

The next step takes us into a new territory. It consists of describing the 4-orientation of the charged particle by L, where L is the q-form of the Lorentz transform which takes the particle from a reference 3-orientation and zero 4-velocity to the observed values of these quantities. From earlier work this means that $V = LL^{\dagger}$. Substituting this in the above gives the following equation for $L(\tau)$:

$$\left(m\dot{L} + \tfrac{1}{2}eq^{\dagger}L\right)L^{\dagger} + L\left(m\dot{L}^{\dagger} + \tfrac{1}{2}eL^{\dagger}q\right) = 0 \tag{39}$$

which implies that

$$\dot{L} = -\frac{e}{2m}q^{\dagger}L + \mathrm{i}X\tilde{L}^{\dagger} \tag{40}$$

where X is some Hermitian quantity.

A particularly natural form for X is motivated by our equation describing a moving particle which is rotating about the z-body axis in its frame of instantaneous rest:

$$X = -\tfrac{1}{2}L\omega(\tau)\sigma^{3}L^{\dagger} \tag{41}$$

which is recognizably Hermitian. This gives

$$\dot{L} = -\frac{e}{2m}q^{\dagger}L - \frac{\mathrm{i}\omega(\tau)}{2}L\sigma^{3}. \tag{42}$$

Comparison with our earlier equation shows that this equation describes the orientation of a body which is rotating with angular velocity $\omega(\tau)\sigma^{3}$ in body axes, and precessing with generalized angular velocity, in space axes, of

$$\boldsymbol{\Omega}\cdot\boldsymbol{\sigma} = -\mathrm{i}\frac{e}{m}q^{\dagger}. \tag{43}$$

In the simpler case in which there is no electric field ($\boldsymbol{E} = 0$), $\boldsymbol{\Omega}$ is real and is given by

$$\boldsymbol{\Omega}\cdot\boldsymbol{\sigma} = -\frac{e}{m}\boldsymbol{H}\cdot\boldsymbol{\sigma} \tag{44}$$

which says that the particle precesses with an angular velocity of $-\frac{e}{m}\boldsymbol{H}$.

This concludes the updating of classical electrodynamics to include a description of a particle which is not simply a featureless point particle but has enough form or structure to make meaningful the definition of an orientation in space.

4. Quantum theory of a spin-1/2 particle in q-notation

In this section we move on to the second stage of our process of linking classical and quantum formulations. This involves doing something new which is to write Dirac's 4-spinors in a quaternionic notation in a way which does not seem previously to have been used. The result of this will be to create a surprisingly close bond between the quantities that we have shown can be useful in describing the orientation of a classical relativistic body and the quantities used since Dirac to describe a spinning relativistic quantum particle.

In place of the six real functions of proper time involved in the quaternion which describes the 4-orientation of the classical electron above, the quantum spin-1/2 particle is, of course, described by four complex fields over space and time. These fields will be written as $\psi_{\alpha}(x)$, where the index α runs from 1 to 4, and x is shorthand for x^{μ}. These fields satisfy Dirac's equation, which can be written in a number of ways, which are equivalent to the following:

$$\left[\gamma^{\mu}(\mathrm{i}\partial_{\mu} - eA_{\mu}) - m\right]\psi = 0 \tag{45}$$

where the matrices γ^{μ} will be chosen in the spinor representation:

$$\gamma^{0} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^{i} = \begin{pmatrix} 0 & -\sigma^{i} \\ \sigma^{i} & 0 \end{pmatrix}. \tag{46}$$

If we write the 4-spinor ψ in terms of two 2-spinors ϕ_i as follows:

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{47}$$

then Dirac's equation can be rewritten as

$$[\mathrm{i}(\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\partial}) - e\tilde{A}]\phi_1 = m\phi_2 \qquad [\mathrm{i}(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\partial}) - eA]\phi_2 = m\phi_1 \tag{48}$$

where A is the quaternion form of the vector potential. Taking the complex conjugate of the second equation gives

$$[-\mathrm{i}(\partial_0 - \boldsymbol{\sigma}^* \cdot \boldsymbol{\partial}) - eA^*]\phi_2^* = m\phi_1^*. \tag{49}$$

If we now introduce the matrix

$$C \equiv -\mathrm{i}\sigma^2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{50}$$

which has the useful property that $C\sigma^{i*}C^{-1} = -\sigma^i$, and multiply the above equation by it, then we find that (since $A^{\mu *} = A^\mu$)

$$[-\mathrm{i}(\partial_0 + \boldsymbol{\sigma} \cdot \boldsymbol{\partial}) - e\tilde{A}]C\phi_2^* = mC\phi_1^*. \tag{51}$$

The next step is the key new idea, which turns out to have many consequences. It is to combine the quantities ϕ_1 and $C\phi_2^*$ into a single complex 2×2 matrix which will be treated as a complex quaternion Q, where

$$Q \equiv (\phi_1, C\phi_2^*). \tag{52}$$

Equations (48) and (51) can then be combined to give the new form of Dirac's equation:

$$\mathrm{i}(\partial_0 + \partial)Q\sigma^3 - e\tilde{A}Q = m\tilde{Q}^\dagger \tag{53}$$

where, in deriving the right-hand side, we have used the general relationship

$$(\phi_2, C\phi_1^*) = (\phi_1, C\phi_2^*)^{\tilde{}\dagger} \tag{54}$$

which can be proved by simple manipulation of the elements of 2×2 matrices.

This form of Dirac's equation is to be compared with other quaternion forms, which since Conway [2] have been chosen to be linear over the complex numbers: which is to say that if Q_1 and Q_2 are solutions then so are $a\,\mathrm{e}^{\mathrm{i}\alpha}Q_1 + b\,\mathrm{e}^{\mathrm{i}\beta}Q_2$, with a, b, α, β real. The above equation is also linear but in the sense that if Q_1 and Q_2 are solutions then so are $Q_1 a \exp(\mathrm{i}\alpha\sigma^3) + Q_2 b \exp(\mathrm{i}\beta\sigma^3)$. It will be shown that there are distinct advantages to this unfamiliar form in the context of finding the classical limit.

It is a standard result that 4-spinors transform under 3-rotations through angle θ about axis i, and under boosts in the i direction to velocity $\tanh(\chi)$ by

$$\psi \to \exp\left(\tfrac{1}{2}\gamma^j\gamma^k\theta\right)\psi \quad \text{and} \quad \psi \to \exp\left(\tfrac{1}{2}\gamma^0\gamma^i\chi\right) \quad \text{respectively.} \tag{55}$$

It is then straightforward to show that with the spinor representation of the γ matrices, the corresponding transformations of Q are given by precisely the forms (3) and (16) we have for the classical particle.

In order to facilitate the proof of the above and related results we give a short dictionary to indicate how various spinor forms translate into quaternion forms.

4.1. Some general relationships between spinors and quaternions

Because of the unfamiliarity of these quaternion forms of spinors it may be useful to summarize some results, which can be proved in a routine manner. The basic identity between a spinor and its q-representation is given above. We now list some further relationships:

If $\psi' =$	then $Q' =$	
$\mathrm{i}\psi$	$\mathrm{i}Q\sigma^3$	
$\mathrm{e}^{\mathrm{i}\alpha}\psi$	$Q\mathrm{e}^{\mathrm{i}\alpha\sigma^3}$	
$\gamma^0\psi$	$\tilde{Q}^\dagger$	
$\gamma^i\psi$	$-\sigma^i\tilde{Q}^\dagger$	
$A_\mu\gamma^\mu\psi$	$A\tilde{Q}^\dagger$	(where A_μ is real and $A \equiv A^0 + A^i\sigma^i$)
$\gamma^\mu\partial_\mu\psi$	$(\partial_0 - \partial)\tilde{Q}^\dagger$	
$A_\mu\gamma^\mu B_\nu\gamma^\nu\psi$	$A\tilde{B}^\dagger Q$	
$\gamma^0\gamma^i\psi$	$\sigma^i Q$	
$\gamma^j\gamma^k\psi$	$\mathrm{i}\sigma^i Q$	
$F_{\mu\nu}\gamma^\mu\gamma^\nu\psi$	FQ	$\left(F_{\mu\nu} = -F_{\nu\mu},\, F = 2(F_{0i} + \mathrm{i}F_{jk})\sigma^i\right)$
$\gamma^5\psi$	$Q\sigma^3$	$(\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3)$
$\exp(\mathrm{i}\alpha\gamma^5)\psi$	$\mathrm{e}^{\mathrm{i}\alpha}Q$	

Note that the spinor forms are generally more complicated and burdened with indices. Also the above translation of the 4-spinors into quaternions gives a very nice form for the current vector J^μ, which is proportional to the familiar $\bar{\psi}\gamma^\mu\psi$. It can be shown that

$$\bar{\psi}\gamma^\mu\psi = 2(QQ^\dagger)^\mu \qquad \text{or equivalently} \qquad 2QQ^\dagger = \sigma^\mu\bar{\psi}\gamma^\mu\psi. \tag{56}$$

Up to a normalizing factor it will be noticed that this form of the 4-current is very similar to that we derived in (19) for the classical rotating particle.

This result should be compared with Gough [7] who uses a different path to obtain a quaternion form of the current: a form which lacks the conciseness of the above because it continues to use explicit spinors.

4.2. Plane wave solutions

We will find that there are still closer correspondences if we look at plane wave solutions of the Dirac equation in the absence of an electromagnetic field. With our representations of the γ matrices, it is a familiar process to establish these solutions. Let us start with the simplest solutions of all, of the form

$$\psi = \psi_0\exp(-\mathrm{i}\omega t) \tag{57}$$

where ψ_0 is constant. Then

$$(\omega\gamma^0 - m)\psi_0 = 0. \tag{58}$$

This linear equation has the following familiar simple solutions:

$$\psi_+ = (\alpha u_1 + \beta u_2)\exp(-\mathrm{i}mt) \qquad \text{or} \qquad \psi_- = (\alpha v_1 + \beta v_2)\exp(\mathrm{i}mt) \tag{59}$$

when $\omega = \pm m$ respectively and α, β are complex numbers, and

$$u_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad u_2 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \qquad v_1 = \begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}. \tag{60}$$

The normal interpretations of these of course are that they represent particle ($\omega = m$) and antiparticle ($\omega = -m$) at rest in states which are linear combinations of states with spins up and down relative to the z-axis. The general state of a particle at rest is a linear combination of these over the complex numbers.

Now we simply take these solutions and translate them into the quaternion notation. They become

$$Q_+ = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \exp(-\mathrm{i}mt\sigma^3) \qquad \text{and} \qquad Q_- = \begin{pmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{pmatrix} \exp(\mathrm{i}mt\sigma^3) \tag{61}$$

when $\omega = m$ and $-m$, respectively. Now $\left(\begin{smallmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{smallmatrix}\right)$ is a real quaternion which in the context of classical physics simply represents a 3-rotation, together with a dilation. So in the case $\omega = m$ we have the solution

$$Q_+ = \lambda R_0 \exp(-\mathrm{i}mt\sigma^3) \tag{62}$$

where $\lambda = \sqrt{|\alpha|^2 + |\beta|^2} > 0$, which in a classical context is, up to the modulus λ, the quaternion describing a body rotating about the z direction in body axes with angular velocity $2m$, and with that axis of rotation in a direction in space axes given by R_0.

In the particular case in which $\beta = 0$ it describes a steady rotation in the positive sense about the z-axis, while the case $\alpha = 0$ gives the case in which the axis has been rotated so that the rotation is about the negative spatial z-axis.

Now these last two statements correspond very well with the usual quantum interpretation of the solutions as representing a particle with spin up and spin down, relative to the z-direction in spatial axes. And linear combinations of the two are taken to represent particles with spins in other directions, also in harmony with the classical spin interpretation.

We are therefore in the happy situation of finding that the mathematical expression describing a classical rotating particle has the same form as the mathematical expression describing the free-field spinor which represents a quantum particle with spin.

Added to this are the new ideas that the quantum phase equates to an angle of rotation about a body axis and that the mass is equal to half the magnitude of the angular velocity. This is of course in natural units in which $c = \hbar = 1$. In usual units this assigns an angular velocity of 1.5527×10^{21} rad s^{-1} to the electron, and of course correspondingly more to other massive fermions.

At first these seem very strange ideas indeed. We are used to phase as being a rather abstract, quantum property not to be seen in classical phenomena. And we are used to thinking of the mass as being a scalar, intrinsic property of a particle. Classically we see no connection between angular velocity and mass. They are quite distinct. And yet it is certainly true that in natural units the mass does have the dimensions of an angular velocity. To the objection that mass is a scalar while angular velocity is a vector it is enough to reply that the mass is only being related to the scalar magnitude of the angular velocity.

Although the connection is far more apparent in the quaternion notation, there is no problem with proving essentially the same thing in spinor notation. By considering how spinors transform under rotations it is quite simple to show that a rotation α about the z-axis gives rise to a phase change of $\alpha/2$, and rotations about more general axes follow.

The conclusion is that there is no reason not to consider the phase and mass in these seemingly strange terms, for a free particle. It may or may not be useful. It certainly gives a new perspective on things, and new perspectives can often lead to other new ideas. But this will not be followed up in this paper.

If we next turn to the solutions with $\omega = -m$ we find that the solution can be written in the form

$$Q_{-} = \mathrm{i}\mu R_0 \exp(\mathrm{i}mt\sigma^3) \tag{63}$$

where $\mu = \sqrt{|\alpha|^2 + |\beta|^2} > 0$.

This differs only in two ways. The first is that it seems to represent a particle which is rotating in the opposite direction about the z-axis in body axes. And the second is the factor of i. The quantum interpretation is that of an antiparticle. The distinguishing feature of the quaternions describing particles and antiparticles is that they have opposite signs to their determinants: note that the determinants are real in both cases. It is immediate from the above definitions that

$$\det(Q_{+}) > 0 \qquad \text{and} \qquad \det(Q_{-}) < 0. \tag{64}$$

The plane wave solutions of the Dirac equation can be derived from those found above for particles at rest by simply boosting them to a general momentum. And a precisely parallel process applies in the quaternion notation.

Thus we have general solutions of the form

$$Q_{+} = \lambda B(p) R \exp\left(-\mathrm{i}p_\mu x^\mu \sigma^3\right) \qquad \text{and} \qquad Q_{-} = \mathrm{i}\mu B(p) R \exp\left(\mathrm{i}p_\mu x^\mu \sigma^3\right) \tag{65}$$

for particle and antiparticle, respectively.

5. The ray optics or classical limit of Dirac's equation

At this stage we have, on one hand, the form (42) of the classical relativistic equation for the 4-orientation of a charged particle in an electromagnetic field, and on the other the quaternionic Dirac equation (53). Each of these equations has as the dependent variable a quantity that transforms in the same way under a Lorentz transformation, though of course one is a function of a scalar variable τ and the other a function of position x^μ.

We now proceed to derive the form of the Dirac equation in the limit of nearly plane waves—the ray optics limit.

First make the following substitution in Dirac's q-equation (53)

$$Q(x) = P(x) \exp(-\mathrm{i}m\Lambda(x)\sigma^3) \tag{66}$$

where both $P(x)$ and $\Lambda(x)$ vary only slightly from unity, if we consider a nearly plane wave corresponding to a particle rather than the antiparticle for which $P(x)$ would be close to i and $\Lambda(x)$ close to -1. Then

$$\mathrm{i}(\partial_0 + \partial)P\sigma^3 + m(\partial_0\Lambda + \partial\Lambda)P - e\tilde{A}P = m\tilde{P}^\dagger. \tag{67}$$

If we further write $U_\mu = \partial_\mu\Lambda - eA_\mu/m$, then this becomes

$$\mathrm{i}(\partial_0 + \partial)P\sigma^3 + m\tilde{U}P = m\tilde{P}^\dagger. \tag{68}$$

Operating on this with $(\partial_0 - \partial)$ gives

$$\mathrm{i}\,\square\,P\sigma^3 + m(\partial_0 - \partial)(\tilde{U}P) = m[(\partial_0 + \partial)P]^{\tilde{}\dagger} \tag{69}$$

which, on using equation (68), becomes

$$\mathrm{i}\,\square\,P\sigma^3 + \square\,\Lambda P + eq^\dagger P + m\left(\tilde{U}\partial_0 P - \sigma^i\tilde{U}\partial_i P\right) = -\mathrm{i}m^2(P - U\tilde{P}^\dagger)\sigma^3. \tag{70}$$

Now we can eliminate the term in $\tilde{P}^\dagger$ from this by using (68) again, to give

$$\mathrm{i}\,\square\,P\sigma^3 + \square\,\Lambda P + eq^\dagger P + 2mU^\mu\partial_\mu P = -\mathrm{i}m^2(1 - U\tilde{U})P\sigma^3 \tag{71}$$

where we have used the identity

$$U(\partial_0 + \partial)P + \left(\tilde{U}\partial_0 P - \sigma^i \tilde{U}\partial_i P\right) = 2S[U(\partial_0 + \partial)P] = 2U^\mu \partial_\mu P. \tag{72}$$

We now proceed to the ray optics limit. This means going to the limit of very high frequency, or $m \to \infty$, which yields the eikonal equation

$$U\tilde{U} \equiv U^\mu U_\mu = 1 \qquad \text{or} \qquad (m\partial^\mu \Lambda - eA^\mu)(m\partial_\mu \Lambda - eA_\mu) = m^2. \tag{73}$$

(For comparison the eikonal equation in ray optics for the electromagnetic field is simply $\partial^\mu \Lambda \partial_\mu \Lambda = 0$.)

This equation is, of course, precisely the classical Hamilton–Jacobi equation.

Given this relationship our equation reduces to

$$\mathrm{i}\,\Box P\sigma^3 + \Box \Lambda P + eq^\dagger P + 2mU^\mu \partial_\mu P = 0. \tag{74}$$

In the same limit the second-order derivatives of Λ and P are small enough to be ignored and we have the simple approximation

$$U^\mu \partial_\mu P + \frac{e}{2m} q^\dagger P = 0. \tag{75}$$

If we now rewrite this in terms of $Q(x) = P(x)\exp(-\mathrm{i}m\Lambda(x)\sigma^3)$, then we find that

$$U^\mu \partial_\mu Q(x) = -\frac{e}{2m} q^\dagger Q(x) - \mathrm{i}\left(m + eU^\mu A_\mu\right) Q(x)\sigma^3. \tag{76}$$

Now consider the system $X(\tau)$ of rays or curves in spacetime parametrized by τ defined by

$$\frac{\mathrm{d}X^\mu(\tau)}{\mathrm{d}\tau} = U^\mu = \partial^\mu \Lambda - \frac{eA^\mu}{m}. \tag{77}$$

Along one of these rays we may define $Q(\tau) = Q(X(\tau))$, so that then

$$\dot{Q}(\tau) = -\frac{e}{2m} q^\dagger Q(\tau) - \mathrm{i}\left(m + eU^\mu A_\mu\right) Q(\tau)\sigma^3. \tag{78}$$

This equation has precisely the same form as equation (42) we have derived for the motion of a rotating charged particle:

$$\dot{L} = -\frac{e}{2m} q^\dagger L - \tfrac{1}{2}\mathrm{i}\omega L\sigma^3. \tag{79}$$

Furthermore, provided that we choose the value of the magnitude of the angular velocity ω of the classical particle to be $2(m + eA \cdot V)$, then the two equations are identical in form.

For the case of the antiparticle the above definition of τ would lead to a trajectory that has the particle moving backwards in time as τ increases. This is a familiar way of looking at antiparticles, of course. If alternatively τ is redefined so that the particle moves forwards in time as τ increases then it is readily shown that

$$\dot{Q}(\tau) = -\frac{(-e)}{2m} q^\dagger Q(\tau) + \mathrm{i}\left(m + (-e)(-U)^\mu A_\mu\right) Q(\tau)\sigma^3 \tag{80}$$

where the term $-U$ has a positive time-like component. This simply describes a particle with charge $-e$ and an opposite sense of spin. This is not surprising in view of the general and well-known result that time inversion is always equivalent to a simultaneous change in sign of charge and parity. In view of this we will avoid lengthy repetitions of results for the antiparticle case in future.

If we let $Q(\tau)Q(\tau)^\dagger = V(\tau)$, then it is readily shown that mod (V) ($V^\mu V_\mu$ in vector notation) will have constant modulus along the ray: the mathematics being identical in form to that used for the motion of a classical charged particle.

The equivalence between these two can now be understood in the following way. The eikonal or Hamilton–Jacobi equation has solutions $\Lambda(x) =$ constant which are surfaces on which the σ^3-phase is constant. For any given solution we can trace a set of curves $X(\tau)$ in spacetime defined by

$$\frac{\mathrm{d}X^\mu(\tau)}{\mathrm{d}\tau} = U^\mu = \partial^\mu\Lambda - \frac{eA^\mu}{m}. \tag{81}$$

Then these curves are precisely the trajectories of the corresponding classical particle, and moreover the 4-velocity of the particle is precisely (up to a normalization factor) given by the value of P at a given point. U is then identifiable with the 4-velocity V of a particle moving along the trajectory.

This means writing the neo-classical equation (42) as

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = -\frac{e}{2m}q^\dagger L - \mathrm{i}(m + eA\cdot V)L\sigma^3. \tag{82}$$

An additional interesting thing about this equation is that the quantity $m + eA\cdot V$, the magnitude of the intrinsic angular velocity, is essentially the Lagrangian that was used to derive the classical equation of motion in the first place. This equates the intrinsic angular velocity with the Lagrangian, and says that the trajectory is therefore such as to make stationary the total angle rotated by the particle about its axis in moving along the path.

This is reminiscent of a property of particle paths in general relativity: they are such as to maximize the proper time on the trajectory.

Note also that this generalizes the idea that the magnitude of the angular velocity is effectively twice the mass m to being twice $m + eA\cdot V$ in an electromagnetic field.

6. Motion of a charged particle in a plane electromagnetic wave

To underline the close connection between our classical solution and the quantum solution we will next solve each equation *exactly*—not in the eikonal limit—in the special case of a plane electromagnetic field, to show how remarkably similar the solutions are.

We will start with the motion of a classical particle. The solution is well known [15]. The main innovation is the introduction of a quaternion notation in order to allow a description of particles with spin. It turns out that the solutions have an extra simplicity of form in this notation.

Then the same situation will be treated in a quantum picture based on Dirac's equation. It is again the case that the solution is well known [12]. But the use of the quaternion rather than spinor description of the field leads to a remarkably close connection between the quantum solutions and those found in the first section for the classical solution. The precise connection is that the quantum solution is a field which is decomposable into a set of classical trajectories in a straightforward way.

6.1. The classical motion

The starting point will be the equation of motion:

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = -\frac{e}{2m}q^\dagger L - \frac{\mathrm{i}\omega}{2}L\sigma^3 \tag{83}$$

in which a particle with mass m, charge e and intrinsic angular velocity $\omega(x^\mu)$ about its axis (taken in the z-direction) which may vary with position x^μ, is moving in an electromagnetic field q, with

$$q = (\partial_0 + \partial)A. \tag{84}$$

The quantity $L(\tau)$ is a quaternion which defines the 4-orientation of the particle and yields the 4-velocity $V(\tau)$ of the particle though the equation $V = LL^\dagger$.

Now the fact that the electromagnetic field is a plane wave means that it can be written in the form

$$A(x) = A(\phi) \qquad \text{where} \quad \phi = k \cdot x. \tag{85}$$

Since A is a solution of Maxwell's equation $\Box A = 0$ we must have

$$k \cdot k = 0. \tag{86}$$

Also, since we have chosen the Lorentz gauge in order for equation (84) to hold,

$$0 = \partial_\mu A^\mu = k_\mu \dot{A}^\mu = k \cdot \dot{A} \tag{87}$$

This means that up to an irrelevant constant we also have

$$k \cdot A = 0. \tag{88}$$

In what follows we will be making frequent use of these identities: $0 = k \cdot k = k \cdot A$, or equivalently $k\tilde{k} = 0$ and $k\tilde{A} + A\tilde{k} = 0$.

Substituting this special form of A in (84) above gives

$$q = \left(k_0 + k_i\sigma^i\right)\dot{A}(\phi) = (k^0 - k^i\sigma^i)\dot{A}(\phi) = \tilde{k}\dot{A} = -\tilde{\dot{A}}k. \tag{89}$$

Hence, since A and k are real vectors,

$$q^\dagger = \dot{A}\tilde{k} = -k\tilde{\dot{A}}. \tag{90}$$

We next show that $k \cdot V$ is constant, since, using

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = \frac{e}{m}(q^\dagger V + Vq) \tag{91}$$

we have, with a little manipulation,

$$\frac{\mathrm{d}(k \cdot V)}{\mathrm{d}\tau} = \frac{e}{m}S[(q^\dagger V + Vq)\tilde{k}] = -\frac{e}{m}S[(k\tilde{\dot{A}}V + V\tilde{\dot{A}}k)\tilde{k}] = 0. \tag{92}$$

Now we will be assuming that the solution L is also a function $L(\phi)$ of ϕ alone, as also will we suppose ω to be. Substituting in equation (83) gives

$$\left(k \cdot \frac{\mathrm{d}x}{\mathrm{d}\tau}\right)\dot{L} = -\frac{e}{2m}\dot{A}\tilde{k}L - \frac{\mathrm{i}\omega}{2}L\sigma^3. \tag{93}$$

Now

$$\mathrm{d}x/\mathrm{d}\tau = V = p/m \tag{94}$$

where p is the 4-momentum of the particle, so, if we let $k \cdot p = \gamma$, which has been proved to be a constant, then the equation of motion becomes simple:

$$2\gamma\dot{L} = -e\dot{A}\tilde{k}L - \mathrm{i}m\omega L\sigma^3. \tag{95}$$

This can be quite easily solved in the form

$$L = \exp\left(-\frac{e}{2\gamma}A\tilde{k}\right)L_0\exp\left(-\frac{\mathrm{i}m}{2\gamma}\int\omega\sigma^3\mathrm{d}\phi\right). \tag{96}$$

The first factor can be simplified as follows. We can expand the exponential in powers of $A\tilde{k}$, but because $A\tilde{k}A\tilde{k} = -A\tilde{A}k\tilde{k} = 0$ (since $k\tilde{k} = 0$), all powers above the first vanish, leaving only

$$L = \left(1 - \frac{e}{2\gamma}A\tilde{k}\right)L_0\exp\left(-\frac{\mathrm{i}m}{2\gamma}\int\omega\sigma^3\mathrm{d}\phi\right). \tag{97}$$

It can easily be shown that this is unimodular if L_0 is unimodular. We can next use this to calculate the 4-velocity $V = LL^\dagger$:

$$V = (1 - \beta A\tilde{k})L_0 L_0^\dagger(1 - \beta\tilde{k}A) \qquad \text{where} \qquad \beta = e/(2\gamma). \tag{98}$$

If $L_0 L_0^\dagger = V_0$ then a little algebra leads to the result

$$mV = mV_0 - eA + \frac{k}{2\gamma}\left[2mV_0 \cdot eA - e^2A^2\right] = mV_0 - eA + \frac{k}{2\gamma}\left[m^2 - (mV_0 - eA)^2\right] \tag{99}$$

or, if we let $mV = p$ and $mV_0 = p_0$,

$$p = p_0 - eA + \frac{k}{2\gamma}\left[m^2 - (p_0 - eA)^2\right]. \tag{100}$$

The interpretation of these results is quite easy. L_0 is the 4-orientation of a particle in the absence of an electromagnetic field, with momentum p_0. Compared with this, the trajectory of a particle in the plane wave is given by L, which is obtained, relative to the field-free case, by a 4-rotation in space axes of $(1 - \beta A\tilde{k})$, and a rotation in body axes of $\exp[-\mathrm{i}m/(2\gamma)\int \omega\sigma^3 \mathrm{d}\phi]$. This results in a momentum p given by (100). Note that the solution in terms of L is both simpler than that in terms of V and also contains scope to describe an intrinsic rotation.

We can also calculate the actual trajectory as follows, from

$$m\frac{\mathrm{d}x}{\mathrm{d}\tau} = p = p_0 - eA + \frac{k}{2\gamma}\left[m^2 - (p_0 - eA)^2\right]. \tag{101}$$

For we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = k \cdot p\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{k \cdot p_0}{m} = \frac{\gamma}{m} \tag{102}$$

consequently the proper time τ and the parameter ϕ differ only by the constant factor γ/m. We can therefore integrate the above equation easily to give

$$x - x_0 = \left[p_0\phi + \int -eA + k[m^2 - (p_0 - eA)^2]/(2\gamma)\mathrm{d}\phi\right] \Big/ \gamma. \tag{103}$$

An alternative and more complicated derivation of this result for the trajectory can be found in [15], working from the Hamilton–Jacobi equation and in the special case of wave motion along the x-axis.

6.2. The quantum motion

Although the results that follow can be obtained in spinor formulations [12] they will be obtained in q-notation here so that parallels with the classical solution will be immediately apparent.

Our starting point will be Dirac's equation in q-notation:

$$\mathrm{i}(\partial_0 + \partial)Q\sigma^3 - e\tilde{A}Q = m\tilde{Q}^\dagger \tag{104}$$

where, as above, $A = A(\phi)$, and $\phi = k \cdot x$, $A \cdot k = k \cdot k = 0$.

In the absence of the electromagnetic field we have seen that plane wave solutions for particles (as opposed to antiparticles) are of the form

$$Q(x) = B_0 \exp\left(-\mathrm{i}p_0 \cdot x\sigma^3\right) \tag{105}$$

where $p_0 = mB_0B_0^\dagger$.

In the presence of the above plane electromagnetic wave, a solution will be sought of the form

$$Q(x) = L(\phi)\exp\left[-\mathrm{i}(p_0 \cdot x + \Lambda(\phi))\sigma^3\right]. \tag{106}$$

Then, remembering that for any function $F(\phi)$, $(\partial_0+\partial)F(\phi) = \tilde{k}\dot{F}$, Dirac's equation becomes

$$\mathrm{i}\left(\tilde{k}\dot{L} - \mathrm{i}(\tilde{p}_0 + \tilde{k}\dot{\Lambda})L\sigma^3\right)\sigma^3 - e\tilde{A}L = m\tilde{L}^\dagger \tag{107}$$

or

$$\mathrm{i}\tilde{k}\dot{L}\sigma^3 + m\tilde{U}L = m\tilde{L}^\dagger \tag{108}$$

where $mU = p_0 + k\dot{\Lambda} - eA$.

We next operate on this equation by $(\partial_0 - \partial)$, noting that generally $(\partial_0 - \partial)F(\phi) = k\dot{F}(\phi)$, so that $(\partial_0 - \partial)(\tilde{k}\dot{L}) = k\tilde{k}\ddot{L} = 0$ and we are left with

$$k(\dot{\tilde{U}L}) = k\dot{\tilde{L}}^\dagger. \tag{109}$$

Eliminating $L^\dagger$ by use of (108) leads to the equation

$$2k\cdot U\dot{L} + k\dot{\tilde{U}}L = -\mathrm{i}m(1-U^2)L\sigma^3. \tag{110}$$

Now $k\cdot U = k\cdot p_0/m = \gamma/m$—the same γ as in the classical section above—and also $m\dot{\tilde{U}} = \tilde{k}\ddot{\Lambda} - e\dot{\tilde{A}}$ so $k\dot{\tilde{U}} = -ek\dot{\tilde{A}}/m$. Hence we have

$$2\gamma\dot{L} = -e\dot{A}\tilde{k}L - \mathrm{i}m^2(1-U^2)L\sigma^3. \tag{111}$$

This is of course the same as the equation that we have met in the classical case, with a minor variation in the last term, and is solved similarly:

$$L = \left(1 - \frac{e}{2\gamma}A\tilde{k}\right)L_0\exp\left(-\frac{\mathrm{i}m}{2\gamma}\int(1-U^2)\sigma^3\,\mathrm{d}\phi\right). \tag{112}$$

Consequently, we have in the quantum case the solution

$$Q(x) = \left(1 - \frac{eA\tilde{k}}{2\gamma}\right)L_0\exp\left[-\mathrm{i}\left(p_0\cdot x + \Lambda(\phi) + \frac{m^2}{2\gamma}\int(1-U^2)\mathrm{d}\phi\right)\sigma^3\right]. \tag{113}$$

But in fact a little algebra shows that

$$\Lambda(\phi) + \frac{m^2}{2\gamma}\int(1-U^2)\,\mathrm{d}\phi = \frac{1}{2\gamma}\int\left[m^2 - (p_0 - eA)^2\right]\mathrm{d}\phi \tag{114}$$

and so

$$Q(x) = \left(1 - \frac{eA\tilde{k}}{2\gamma}\right)L_0\exp\left[-\mathrm{i}\left(p_0\cdot x + \frac{1}{2\gamma}\int\left[m^2 - (p_0-eA)^2\right]\mathrm{d}\phi\right)\sigma^3\right]. \tag{115}$$

Note that if we wish we may use the freedom of choice in Λ to make $U = V$, and $U^2 = 1$—the eikonal equation—by letting

$$\dot{\Lambda} = \frac{k}{2\gamma}[m^2 - (p - eA)^2]. \tag{116}$$

We next note that L_0 is not totally free, for if we substitute back into Dirac's equation we find, after a little algebra, that

$$\left(\tilde{p}_0 - e\tilde{A} + \tilde{k}\left[m^2 - (p_0 - eA)^2\right]\right) = m(1 - e\tilde{A}k/2\gamma)\tilde{L}_0^\dagger L_0^{-1}(1 - eA\tilde{k}/2\gamma). \tag{117}$$

Then by taking the quaternion conjugate of the above and following the same path as in the classical case, we get the result

$$mV = mV_0 - eA + \frac{k}{2\gamma}\left[m^2 - (mV_0 - eA)^2\right] \tag{118}$$

or, with $mV = p$ and $mV_0 = p_0$

$$p = p_0 - eA + \frac{k}{2\gamma}\left[m^2 - (p_0 - eA)^2\right] \tag{119}$$

which is of course the same as the classical form (100).

7. Comparisons

We now have the solutions to the classical equation and the quantum equation in remarkably similar forms. The difference lies in the rotation about the intrinsic axis, which corresponds to a phase change in spinor notation.

In the quantum case we have the rotation

$$\exp\left[-\mathrm{i}\left(p_0 \cdot x + \frac{1}{2\gamma}\int\left[m^2 - (p_0 - eA)^2\right]\mathrm{d}\phi\right)\sigma^3\right] \tag{120}$$

and in the classical case we have the rotation

$$\exp\left(-\frac{\mathrm{i}m}{2\gamma}\int \omega\sigma^3\,\mathrm{d}\phi\right). \tag{121}$$

In the classical formulation, the quantity ω is undefined: there is nothing in the classical theory to restrict it. Consequently we can now equate the above two expressions to define ω. This gives (recall $\gamma\,\mathrm{d}\tau = m\,\mathrm{d}\phi$)

$$m\omega = 2mp_0\,\mathrm{d}x/\mathrm{d}\tau + \left[m^2 - (p_0 - eA)^2\right]. \tag{122}$$

This is not very satisfactory since it involves the arbitrary constant p_0. But we have the fact that the generalized momentum $p + eA$ satisfies

$$p + eA = p_0 + \left[m^2 - (p_0 - eA)^2\right]k/2\gamma \tag{123}$$

so, in the classical case we can write

$$(p + eA)\,\mathrm{d}x = p_0\,\mathrm{d}x + \left[m^2 - (p_0 - eA)^2\right]k\,\mathrm{d}x/2\gamma \tag{124}$$

$$= p_0 \cdot \mathrm{d}x + \left[m^2 - (p_0 - eA)^2\right]\mathrm{d}\tau/2\gamma. \tag{125}$$

We can then use this to eliminate p_0 from (122) to give

$$m\omega = 2m\,(p + eA)\,\mathrm{d}x \tag{126}$$

so our modified classical equation would then become

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = -\frac{e}{2m}q^{\dagger}L - \mathrm{i}(p + eA)\cdot\frac{\mathrm{d}x}{\mathrm{d}\tau}L\sigma^3. \tag{127}$$

But since $p = mV$ and $\mathrm{d}x/\mathrm{d}\tau = V$, this can be rewritten as

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = -\frac{e}{2m}q^{\dagger}L - \mathrm{i}(m + A\cdot V)L\sigma^3 \tag{128}$$

which is of course the neo-classical equation (82) derived earlier in the ray optics approximation and has here been recovered *exactly* in the case of movement in a plane wave.

8. Conclusion

We have shown that provided we model a charged fermion with mass m in an electromagnetic vector potential A as a particle rotating with intrinsic angular velocity $2(m + eA \cdot V)$, then there is a very smooth and immediate correspondence between such a neo-classical model and that provided by Dirac's equation in both the ray-optics limit or the case of motion in a plane electromagnetic wave. The corollary is that the classical motion is such as to make stationary the total angle rotated by such a particle along its trajectory. This neo-classical equation of motion for a spinning charged particle could be of practical value in experimental situations where the charge and spin of a fermion are relevant but interference and other quantum effects are not.

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